

FACTORIZATION PROPERTIES AND TOPOLOGICAL CENTERS OF LEFT MODULE ACTIONS

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ABSTRACT. For a Banach left module action, we will extend some propositions from Lau and Ülger and others into general situations and we establish the relationships between topological centers of the left module action with the multiplier and factorization properties of left module actions. We have some applications in the dual groups.

1. Introduction and Preliminaries

As is well-known [1], the second dual A^{**} of A endowed with the either Arens multiplications is a Banach algebra. The constructions of the two Arens multiplications in A^{**} lead us to definition of topological centers for A^{**} with respect both Arens multiplications. The topological centers of Banach algebras, module actions and applications of them were introduced and discussed in [6, 8, 13, 14, 15, 16, 17, 21, 22], and they have attracted by some attentions.

Now we introduce some notations and definitions that we used throughout this paper. Let A be a Banach algebra. We say that a net $(e_\alpha)_{\alpha \in I}$ in A is a left approximate identity ($= LAI$) [resp. right approximate identity ($= RAI$)] if, for each $a \in A$, $e_\alpha a \rightarrow a$ [resp. $ae_\alpha \rightarrow a$]. For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by $\langle a'a, b \rangle = \langle a', ab \rangle = a'(ab)$ and $\langle aa', b \rangle = \langle a', ba \rangle = a'(ba)$ for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a, a' \rangle = \langle a', a \rangle$ for every $a \in A$ and $a' \in A^*$. We denote the set $\{a'a : a \in A \text{ and } a' \in A^*\}$ and $\{aa' : a \in A \text{ and } a' \in A^*\}$ by A^*A and AA^* , respectively, clearly these two sets are subsets of A^* . Let A has a BAI . If the equality $A^*A = A^*$, ($AA^* = A^*$) holds, then we say that A^* factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that A^* factors on both sides. Let X, Y, Z be normed spaces and $m : X \times Y \rightarrow Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as following:

1. $m^* : Z^* \times X \rightarrow Y^*$, given by $\langle m^*(z', x), y \rangle = \langle z', m(x, y) \rangle$ where $x \in X$, $y \in Y$, $z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \rightarrow X^*$, given by $\langle m^{**}(y'', z'), x \rangle = \langle y'', m^*(z', x) \rangle$ where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \rightarrow Z^{**}$, given by $\langle m^{***}(x'', y''), z' \rangle = \langle x'', m^{**}(y'', z') \rangle$ where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.

The mapping m^{***} is the unique extension of m such that $x'' \rightarrow m^{***}(x'', y'')$ from X^{**} into Z^{**} is *weak* - to - weak** continuous for every $y'' \in Y^{**}$, but the mapping

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$y'' \rightarrow m^{***}(x'', y'')$ is not in general *weak*-to-weak** continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{x'' \in X^{**} : y'' \rightarrow m^{***}(x'', y'') \text{ is } weak^* - to - weak^* - \text{continuous}\}.$$

Let now $m^t : Y \times X \rightarrow Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z , and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \rightarrow Z^{**}$. The mapping $m^{t***} : X^{**} \times Y^{**} \rightarrow Z^{**}$ in general is not equal to m^{***} , see [1], if $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \rightarrow m^{t***t}(x'', y'')$ is *weak*-to-weak** continuous for every $y'' \in Y^{**}$, but the mapping $x'' \rightarrow m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general *weak*-to-weak** continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : x'' \rightarrow m^{t***t}(x'', y'') \text{ is } weak^* - to - weak^* - \text{continuous}\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j) \rangle = \lim_j \lim_i \langle z', m(x_i, y_j) \rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [6, 18].

The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**} , the second dual of A . By *Goldstine's* Theorem [6, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_\alpha \lim_\beta \langle a', m(a_\alpha, b_\beta) \rangle = \langle a''b'', a' \rangle$$

and

$$\lim_\beta \lim_\alpha \langle a', m(a_\alpha, b_\beta) \rangle = \langle a''ob'', a' \rangle,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**} , respectively, see [6, 14, 18].

The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.

Let A and B be normed spaces. In this paper, if T is a continuous linear operator from A into B , then we write $T \in \mathbf{B}(A, B)$.

This paper is organized as follows.

A. Let B be a Banach left A -module and $(e_\alpha)_\alpha \subseteq A$ be a *LBAI* for B . Then the following assertions hold.

- (1) For each $b' \in B^*$, $\pi_\ell^*(b', e_\alpha) \xrightarrow{w^*} b'$.
- (2) B^* factors on the left with respect to A if and only if B^{**} has a W^*LBAI $(e_\alpha)_\alpha \subseteq A$.
- (3) B^{**} has a W^*LBAI $(e_\alpha)_\alpha \subseteq A$ if and only if B^{**} has a left unit element $e'' \in A^{**}$ such that $e_\alpha \xrightarrow{w^*} e''$.

B. Let B be a Banach left A –module and suppose that $b' \in \text{wap}_\ell(B)$. Let $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \xrightarrow{w^*} a''$ in A^{**} . Then we have

$$\pi_\ell^*(b', a_\alpha) \xrightarrow{w} \pi_\ell^{****}(b', a'').$$

C. Let B be a Banach left A –module and B^* factors on the left with respect to A . If $AA^{**} \subseteq Z_{B^{**}}(A^{**})$, then $Z_{B^{**}}(A^{**}) = A^{**}$.

D. Let B be a Banach left A –module and B^{**} has a $LBAI$ with respect to A^{**} . Then B^{**} has a left unit with respect to A^{**} .

E. Let B be a Banach left A –module and it has a $LBAI$ with respect to A . Then we have the following assertions.

- (1) B^* factors on the left with respect to A if and only if for each $b' \in B^*$, we have $\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b'$ in B^* .
- (2) B factors on the left with respect to A if and only if for each $b \in B$, we have $\pi_\ell^*(b, e_\alpha) \xrightarrow{w} b$ in B .

F. Let B be a Banach left A –module and A has a $LBAI$ $(e_\alpha)_\alpha \subseteq A$ such that $e_\alpha \xrightarrow{w^*} e''$ in A^{**} where e'' is a left unit for A^{**} . Suppose that $Z_{e''}^t(B^{**}) = B^{**}$. Then, B factors on the right with respect to A if and only if e'' is a left unit for B^{**} .

G. Let B be a left Banach A –module and $T \in \mathbf{B}(A, B)$. Consider the following statements.

- (1) $T = \ell_b$, for some $b \in B$.
- (2) $T^{**}(a'') = \pi_\ell^{***}(a'', b'')$ for some $b'' \in B^{**}$ such that $\tilde{Z}_{b''}(A^{**}) = A^{**}$.
- (3) $T^*(B^*) \subseteq BB^*$.

Then (1) \Rightarrow (2) \Rightarrow (3).

If we take $T \in RM(A, B)$ and if B has a sequential BAI and it is WSC , then (1), (2) and (3) are equivalent.

2. Factorization properties and topological centers of left module actions

Let B be a Banach A –bimodule, and let

$$\pi_\ell : A \times B \rightarrow B \text{ and } \pi_r : B \times A \rightarrow B.$$

be the left and right module actions of A on B . Then B^{**} is a Banach A^{**} –bimodule with module actions

$$\pi_\ell^{***} : A^{**} \times B^{**} \rightarrow B^{**} \text{ and } \pi_r^{***} : B^{**} \times A^{**} \rightarrow B^{**}.$$

Similarly, B^{**} is a Banach A^{**} –bimodule with module actions

$$\pi_\ell^{t****} : A^{**} \times B^{**} \rightarrow B^{**} \text{ and } \pi_r^{t****} : B^{**} \times A^{**} \rightarrow B^{**}.$$

We may therefore define the topological centers of the right and left module actions of A on B as follows:

$$\begin{aligned} Z_{A^{**}}(B^{**}) &= Z(\pi_r) = \{b'' \in B^{**} : \text{the map } a'' \rightarrow \pi_r^{***}(b'', a'') : A^{**} \rightarrow B^{**} \\ &\quad \text{is weak}^* \text{--to--weak}^* \text{ continuous}\} \\ Z_{B^{**}}(A^{**}) &= Z(\pi_\ell) = \{a'' \in A^{**} : \text{the map } b'' \rightarrow \pi_\ell^{***}(a'', b'') : B^{**} \rightarrow B^{**} \end{aligned}$$

is weak - to - weak* continuous*

$$Z_{A^{**}}^t(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} : \text{the map } a'' \rightarrow \pi_\ell^{t***}(b'', a'') : A^{**} \rightarrow B^{**}$$

is weak - to - weak* continuous*

$$Z_{B^{**}}^t(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} : \text{the map } b'' \rightarrow \pi_r^{t***}(a'', b'') : B^{**} \rightarrow B^{**}$$

is weak - to - weak* continuous*

We note also that if B is a left (resp. right) Banach A -module and $\pi_\ell : A \times B \rightarrow B$ (resp. $\pi_r : B \times A \rightarrow B$) is left (resp. right) module action of A on B , then B^* is a right (resp. left) Banach A -module.

We write $ab = \pi_\ell(a, b)$, $ba = \pi_r(b, a)$, $\pi_\ell(a_1 a_2, b) = \pi_\ell(a_1, a_2 b)$, $\pi_r(b, a_1 a_2) = \pi_r(b a_1, a_2)$, $\pi_\ell^*(a_1 b', a_2) = \pi_\ell^*(b', a_2 a_1)$, $\pi_r^*(b' a, b) = \pi_r^*(b', ab)$, for all $a_1, a_2, a \in A$, $b \in B$ and $b' \in B^*$ when there is no confusion.

Let B be a left Banach A -module and e be a left unit element of A . Then we say that e is a left unit (resp. weakly left unit) A -module for B , if $\pi_\ell(e, b) = b$ (resp. $\langle b', \pi_\ell(e, b) \rangle = \langle b', b \rangle$ for all $b' \in B^*$) where $b \in B$. The definition of right unit (resp. weakly right unit) A -module is similar.

We say that a Banach A -bimodule B is a unital A -module, if B has left and right unit A -module that are equal then we say that B is unital A -module.

Let B be a left Banach A -module and $(e_\alpha)_\alpha \subseteq A$ be a LAI [resp. weakly left approximate identity (=WLAI)] for A . We say that $(e_\alpha)_\alpha$ is left approximate identity (=LAI) [resp. weakly left approximate identity (=WLAI)] for B , if for all $b \in B$, we have $\pi_\ell(e_\alpha, b) \rightarrow b$ (resp. $\pi_\ell(e_\alpha, b) \xrightarrow{w} b$). The definition of the right approximate identity (=RAI) [resp. weakly right approximate identity (=WRAI)] is similar.

We say that $(e_\alpha)_\alpha$ is a approximate identity (=AI) [resp. weakly approximate identity (WAI)] for B , if B has the same left and right approximate identity [resp. weakly left and right approximate identity].

Let $(e_\alpha)_\alpha \subseteq A$ be weak* left approximate identity for A^{**} . We say that $(e_\alpha)_\alpha$ is weak* left approximate identity A^{**} -module (=W*LAI A^{**} -module) for B^{**} , if for all $b'' \in B^{**}$, we have $\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w^*} b''$. The definition of the weak* right approximate identity A^{**} -module (=W*RAI A^{**} -module) is similar.

We say that $(e_\alpha)_\alpha$ is a weak* approximate identity A^{**} -module (=W*AI A^{**} -module) for B^{**} , if B^{**} has the same weak* left and right approximate identity A^{**} -module.

Let B be a Banach A -bimodule. We say that B is a left [resp. right] factors with respect to A , if $BA = B$ [resp. $AB = B$].

Theorem 2-1. Let B be a Banach left A -module and $(e_\alpha)_\alpha \subseteq A$ be a LBAI for B . Then the following assertions hold.

- (1) For each $b' \in B^*$, we have $\pi_\ell^*(b', e_\alpha) \xrightarrow{w^*} b'$.
- (2) B^* factors on the left with respect to A if and only if B^{**} has a W*LBAI $(e_\alpha)_\alpha \subseteq A$.
- (3) B^{**} has a W*LBAI $(e_\alpha)_\alpha \subseteq A$ if and only if B^{**} has a left unit element $e'' \in A^{**}$ such that $e_\alpha \xrightarrow{w^*} e''$.

Proof. (1) Let $b \in B$ and $b' \in B^*$. Since $|\langle b', \pi_\ell(e_\alpha, b) \rangle| \leq \|b'\| \|\pi_\ell(e_\alpha, b)\|$, we have the following equality

$$\lim_\alpha \langle \pi_\ell^*(b', e_\alpha), b \rangle = \lim_\alpha \langle b', \pi_\ell(e_\alpha, b) \rangle = 0.$$

It follows that $\pi_\ell^*(b', e_\alpha) \xrightarrow{w^*} 0$.

(2) Let B^* factors on the left with respect to A . Then for every $b' \in B^*$, there are $x' \in B^*$ and $a \in A$ such that $b' = x'a$. Then for every $b'' \in B^{**}$, we have

$$\begin{aligned} \langle \pi_\ell^{***}(e_\alpha, b''), b' \rangle &= \langle e_\alpha, \pi_\ell^{**}(b'', b') \rangle = \langle \pi_\ell^{**}(b'', b'), e_\alpha \rangle \\ &= \langle b'', \pi_\ell^*(b', e_\alpha) \rangle = \langle b'', \pi_\ell^*(x'a, e_\alpha) \rangle = \langle b'', \pi_\ell^*(x', ae_\alpha) \rangle \\ &= \langle \pi_\ell^{**}(b'', x'), ae_\alpha \rangle \rightarrow \langle \pi_\ell^{**}(b'', x'), a \rangle \\ &= \langle b'', b' \rangle. \end{aligned}$$

It follows that

$$\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w^*} b'',$$

and so B^{**} has W^*LBAI .

Conversely, let $b' \in B^*$. Then for every $b'' \in B^{**}$, we have

$$\langle b'', \pi_\ell^*(b', e_\alpha) \rangle = \langle \pi_\ell^{***}(e_\alpha, b''), b' \rangle \rightarrow \langle b'', b' \rangle.$$

It follows that

$$\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b',$$

and so by Cohen factorization theorem, we are done.

(3) Assume that B^{**} has a W^*LBAI $(e_\alpha)_\alpha \subseteq A$. Without loss generality, let $e'' \in A^{**}$ be a left unit for A^{**} with respect to the first Arens product such that $e_\alpha \xrightarrow{w^*} e''$. Then for each $b' \in B^*$, we have

$$\begin{aligned} \langle \pi_\ell^{***}(e'', b''), b' \rangle &= \langle e'', \pi_\ell^{**}(b'', b') \rangle \\ &= \lim_\alpha \langle e_\alpha, \pi_\ell^{**}(b'', b') \rangle = \lim_\alpha \langle \pi_\ell^{**}(b'', b'), e_\alpha \rangle \\ &= \lim_\alpha \langle b'', \pi_\ell^*(b', e_\alpha) \rangle = \lim_\alpha \langle \pi_\ell^{***}(b', e_\alpha), b'' \rangle \\ &= \lim_\alpha \langle b', \pi_\ell^{***}(e_\alpha, b'') \rangle = \lim_\alpha \langle \pi_\ell^{***}(e_\alpha, b''), b' \rangle \\ &= \langle b'', b' \rangle. \end{aligned}$$

Thus $e'' \in A^{**}$ is a left unit for B^{**} .

Conversely, let $e'' \in A^{**}$ be a left unit for B^{**} and assume that $e_\alpha \xrightarrow{w^*} e''$ in A^{**} . Then for every $b'' \in B^{**}$ and $b' \in B^*$, we have

$$\begin{aligned} \langle \pi_\ell^{***}(e_\alpha, b''), b' \rangle &= \langle e_\alpha, \pi_\ell^{**}(b'', b') \rangle \\ &\rightarrow \langle e'', \pi_\ell^{**}(b'', b') \rangle = \langle \pi_\ell^{***}(e'', b''), b' \rangle \\ &= \langle b'', b' \rangle. \end{aligned}$$

It follows that $\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w^*} b''$.

□

Corollary 2-2. Let B be a Banach left A – module and A has a $BLAI$. If B^{**} has a W^*LBAI , then

$$\{a'' \in A^{**} : Aa'' \subseteq A\} \subseteq Z_{B^{**}}(A^{**}).$$

Proof. By using the proceeding theorem, since B^{**} has W^*LBAI , B^* factors on the left with respect to A . Suppose that $b' \in B^*$. Then there are $x' \in B^*$ and $a \in A$ such that $b' = x'a$. Assume that $a'' \in A^{**}$ such that $Aa'' \subseteq A$. Let $b'' \in B^{**}$ and $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$ in B^{**} . Then we have the following equality

$$\begin{aligned} \lim_\alpha \langle \pi_\ell^{***}(a'', b''_\alpha), b' \rangle &= \lim_\alpha \langle \pi_\ell^{***}(a'', b''_\alpha), x'a \rangle \\ &= \lim_\alpha \langle a\pi_\ell^{***}(a'', b''_\alpha), x' \rangle = \langle \pi_\ell^{***}(aa'', b''), x' \rangle = \langle \pi_\ell^{***}(a'', b''), b' \rangle. \end{aligned}$$

It follows that $a'' \in Z_{B^{**}}(A^{**})$. □

In the proceeding corollary, if we take $B = A$, then we have the following conclusion

$$\{a'' \in A^{**} : Aa'' \subseteq A\} \subseteq Z_1(A^{**}).$$

A functional a' in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a \rightarrow a'a$ from A into A^* is weakly compact. The proceeding definition is equivalent to the following condition, see [6, 14, 18].

For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a \in A : \|a\| \leq 1\}$, we have

$$\lim_\alpha \lim_\beta \langle a', a_\alpha b_\beta \rangle = \lim_\beta \lim_\alpha \langle a', a_\alpha b_\beta \rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on A is denoted by $wap(A)$. Also we have $a' \in wap(A)$ if and only if $\langle a''b'', a' \rangle = \langle a''ob'', a' \rangle$ for every $a'', b'' \in A^{**}$.

Let B be a Banach left A – module. Then, $b' \in B^*$ is said to be left weakly almost periodic functional if the set $\{\pi_\ell(b', a) : a \in A, \|a\| \leq 1\}$ is relatively weakly compact. We denote by $wap_\ell(B)$ the closed subspace of B^* consisting of all the left weakly almost periodic functionals in B^* .

The definition of the right weakly almost periodic functional ($= wap_r(B)$) is the same.

By [18], $b' \in wap_\ell(B)$ if and only if

$$\langle \pi_\ell^{***}(a'', b''), b' \rangle = \langle \pi_\ell^{t***t}(a'', b''), b' \rangle$$

for all $a'' \in A^{**}$ and $b'' \in B^{**}$. Thus, we can write

$$\begin{aligned} wap_\ell(B) &= \{b' \in B^* : \langle \pi_\ell^{***}(a'', b''), b' \rangle = \langle \pi_\ell^{t***t}(a'', b''), b' \rangle \\ &\quad \text{for all } a'' \in A^{**}, b'' \in B^{**}\}. \end{aligned}$$

Theorem 2-3. Let B be a Banach left A – module and suppose that $b' \in wap_\ell(B)$.

Let $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \xrightarrow{w^*} a''$ in A^{**} . Then we have

$$\pi_\ell^*(b', a_\alpha) \xrightarrow{w} \pi_\ell^{t***t}(b', a'').$$

Proof. Assume that $b'' \in B^{**}$. Then we have the following equality

$$\begin{aligned} \langle \pi_\ell^{****}(b', a''), b'' \rangle &= \langle \pi_\ell^{***}(a'', b''), b' \rangle = \lim_\alpha \langle \pi_\ell^{***}(a_\alpha, b''), b' \rangle \\ &= \lim_\alpha \langle b'', \pi_\ell^*(b', a_\alpha) \rangle. \end{aligned}$$

Now suppose that $(b''_\beta)_\beta \subseteq B^{**}$ such that $b''_\beta \xrightarrow{w^*} b''$. Since $b' \in \text{wap}_\ell(B)$, we have

$$\begin{aligned} \langle \pi_\ell^{****}(b', a''), b''_\beta \rangle &= \langle \pi_\ell^{***}(a'', b''_\beta), b' \rangle \rightarrow \langle \pi_\ell^{***}(a'', b''), b' \rangle \\ &= \langle \pi_\ell^{****}(b', a''), b'' \rangle. \end{aligned}$$

Thus $\pi_\ell^{****}(b', a'') \in (B^{**}, \text{weak}^*)^* = B^*$. So we conclude that

$$\pi_\ell^*(b', a_\alpha) \xrightarrow{w} \pi_\ell^{****}(b', a'') \text{ in } B^{**}.$$

□

In the proceeding corollary, if we take $B = A$, then we obtain the following result. Suppose that $a' \in \text{wap}(A)$ and $a'' \in A^{**}$ such that $a_\alpha \xrightarrow{w^*} a''$ where $(a_\alpha)_\alpha \subseteq A$. Then we have $a'a_\alpha \xrightarrow{w} a'a''$.

Theorem 2-4. Let B be a Banach left A – module and it has a BLAI $(e_\alpha)_\alpha \subseteq A$. Suppose that $b' \in \text{wap}_\ell(B)$. Then we have

$$\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b'.$$

Proof. Let $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$ in B^{**} . Then for every $b' \in \text{wap}_\ell(B)$, we have the following equality

$$\begin{aligned} \lim_\alpha \langle b'', \pi_\ell^*(b', e_\alpha) \rangle &= \lim_\alpha \langle \pi_\ell^{****}(b', e_\alpha), b'' \rangle \\ &= \lim_\alpha \langle b', \pi_\ell^{***}(e_\alpha, b'') \rangle = \lim_\alpha \langle \pi_\ell^{***}(e_\alpha, b''), b' \rangle \\ &= \lim_\alpha \lim_\beta \langle \pi_\ell(e_\alpha, b_\beta), b' \rangle = \lim_\beta \lim_\alpha \langle \pi_\ell(e_\alpha, b_\beta), b' \rangle \\ &= \lim_\beta \langle b_\beta, b' \rangle = \langle b'', b' \rangle. \end{aligned}$$

It follows that

$$\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b'.$$

□

Corollary 2-5. Let B be a Banach left A – module and it has a BLAI $(e_\alpha)_\alpha \subseteq A$. Suppose that $\text{wap}_\ell(B) = B^*$. Then B^* factors on the left with respect to A .

Corollary 2-6. Let A be an Arens regular Banach algebra with LBAI. Then A^* factors on the left.

Example 2-7. i) Let G be finite group. Then we have the following equality

$$M(G)^* L^1(G) = M(G)^* \text{ and } L^\infty(G) L^1(G) = L^\infty(G).$$

ii) Consider the Banach algebra (ℓ^1, \cdot) that is Arens regular Banach algebra with unit element. Then we have $\ell^\infty \cdot \ell^1 = \ell^\infty$.

Theorem 2-8. Let B be a Banach left A -module and B^* factors on the left with respect to A . If $AA^{**} \subseteq Z_{B^{**}}(A^{**})$, then $Z_{B^{**}}(A^{**}) = A^{**}$.

Proof. Let $b'' \in B^{**}$ and $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$ in B^{**} . Suppose that $a'' \in A^{**}$. Since B^* factors on the left with respect to A , for every $b' \in B^*$, there are $x' \in B^*$ and $a \in A$ such that $b' = x'a$. Since $aa'' \in Z_{B^{**}}(A^{**})$, we have

$$\begin{aligned} \langle \pi_\ell^{***}(a'', b''_\alpha), b' \rangle &= \langle \pi_\ell^{***}(a'', b''_\alpha), x'a \rangle \\ &= \langle a\pi_\ell^{***}(a'', b''_\alpha), x' \rangle = \langle \pi_\ell^{***}(aa'', b''_\alpha), x' \rangle \\ &\rightarrow \langle \pi_\ell^{***}(aa'', b''), x' \rangle = \langle \pi_\ell^{***}(a'', b''), b' \rangle. \end{aligned}$$

It follows that

$$\pi_\ell^{***}(a'', b''_\alpha) \xrightarrow{w^*} \pi_\ell^{***}(a'', b''),$$

and so $a'' \in Z_{B^{**}}(A^{**})$. □

Corollary 2-9. Let A be a Banach algebra and A^* factors on the left. If $AA^{**} \subseteq Z_1(A^{**})$, then A is Arens regular.

Theorem 2-10. Let B be a Banach left A -module and B^{**} has a $LBAI$ with respect to A^{**} . Then B^{**} has a left unit with respect to A^{**} .

Proof. Assume that $(e''_\alpha)_\alpha \subseteq A^{**}$ is a $LBAI$ for B^{**} . By passing to a subnet, we may suppose that there is $e'' \in A^{**}$ such that $e''_\alpha \xrightarrow{w^*} e''$ in A^{**} . Then for every $b'' \in B^{**}$ and $b' \in B^*$, we have

$$\begin{aligned} \langle \pi_\ell^{***}(e'', b''), b' \rangle &= \langle e'', \pi_\ell^{**}(b'', b') \rangle = \lim_\alpha \langle e''_\alpha, \pi_\ell^{**}(b'', b') \rangle \\ &= \lim_\alpha \langle \pi_\ell^{***}(e''_\alpha, b''), b' \rangle = \langle b'', b' \rangle. \end{aligned}$$

It follows that $\pi_\ell^{***}(e'', b'') = b''$. □

Corollary 2-11. Let A be a Banach algebra and A^{**} has a $LBAI$. Then A^{**} has a left unit with respect to the first Arens product.

Theorem 2-12. Let B be a Banach left A -module and it has a $LBAI$ with respect to A . Then we have the following assertions.

- (1) B^* factors on the left with respect to A if and only if for each $b' \in B^*$, we have $\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b'$ in B^* .

- (2) B factors on the left with respect to A if and only if for each $b \in B$, we have $\pi_\ell^*(b, e_\alpha) \xrightarrow{w} b$ in B .

Proof. (1) Assume that B^* factors on the left with respect to A . Then for every $b' \in B^*$, there are $x' \in B^*$ and $a \in A$ such that $b' = x'a$. Then for every $b'' \in B^{**}$, we have

$$\begin{aligned} \langle b'', \pi_\ell^*(b', e_\alpha) \rangle &= \langle b'', \pi_\ell^*(x'a, e_\alpha) \rangle = \langle b'', \pi_\ell^*(x', ae_\alpha) \rangle \\ &= \langle \pi_\ell^{**}(b'', x'), ae_\alpha \rangle \rightarrow \langle \pi_\ell^{**}(b'', x'), a \rangle \\ &= \langle b'', b' \rangle. \end{aligned}$$

It follows that $\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b'$ in B^* . The converse by Cohen factorization theorem hold.

- (2) It is similar to the proceeding proof. □

In the proceeding theorem, if we take $B = A$, we obtain Lemma 2.1 from [14].

Let B be a Banach A -bimodule and $a'' \in A^{**}$. We define the locally topological centers of the left and right module actions of a'' on B , respectively, as follows

$$\begin{aligned} Z_{a''}^t(B^{**}) &= Z_{a''}^t(\pi_\ell^t) = \{b'' \in B^{**} : \pi_\ell^{t***t}(a'', b'') = \pi_\ell^{***}(a'', b'')\}, \\ Z_{a''}(B^{**}) &= Z_{a''}(\pi_r^t) = \{b'' \in B^{**} : \pi_r^{t***t}(b'', a'') = \pi_r^{***}(b'', a'')\}. \end{aligned}$$

It is clear that

$$\bigcap_{a'' \in A^{**}} Z_{a''}^t(B^{**}) = Z_{A^{**}}^t(B^{**}) = Z(\pi_\ell^t),$$

$$\bigcap_{a'' \in A^{**}} Z_{a''}(B^{**}) = Z_{A^{**}}(B^{**}) = Z(\pi_r).$$

The definition of $Z_{b''}^t(A^{**})$ and $Z_{b''}(A^{**})$ for some $b'' \in B^{**}$ are the same.

Theorem 2-13. Let B be a Banach left A -module and A has a LBAI $(e_\alpha)_\alpha \subseteq A$ such that $e_\alpha \xrightarrow{w^*} e''$ in A^{**} where e'' is a left unit for A^{**} . Suppose that $Z_{e''}^t(B^{**}) = B^{**}$. Then, B factors on the right with respect to A if and only if e'' is a left unit for B^{**} .

Proof. Assume that B factors on the right with respect to A . Then for every $b \in B$, there are $x \in B$ and $a \in A$ such that $b = ax$. Then for every $b' \in B^*$, we have

$$\begin{aligned} \langle \pi_\ell^*(b', e_\alpha), b \rangle &= \langle b', \pi_\ell(e_\alpha, b) \rangle = \langle \pi_\ell^{***}(e_\alpha, b), b' \rangle \\ &= \langle \pi_\ell^{***}(e_\alpha, ax), b' \rangle = \langle \pi_\ell^{***}(e_\alpha a, x), b' \rangle \\ &= \langle e_\alpha a, \pi_\ell^{**}(x, b') \rangle = \langle \pi_\ell^{**}(x, b'), e_\alpha a \rangle \\ &\rightarrow \langle \pi_\ell^{**}(x, b'), a \rangle = \langle b', b \rangle. \end{aligned}$$

It follows that $\pi_\ell^*(b', e_\alpha) \xrightarrow{w^*} b'$ in B^* . Let $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$ in B^{**} . Since $Z_{e''}^t(B^{**}) = B^{**}$, for every $b' \in B^*$, we have the following equality

$$\begin{aligned} \langle \pi_\ell^{***}(e'', b''), b' \rangle &= \lim_{\alpha} \lim_{\beta} \langle b', \pi_\ell(e_\alpha, b_\beta) \rangle \\ &= \lim_{\beta} \lim_{\alpha} \langle b', \pi_\ell(e_\alpha, b_\beta) \rangle = \lim_{\beta} \langle b', b_\beta \rangle \\ &= \langle b'', b' \rangle. \end{aligned}$$

It follows that $\pi_\ell^{***}(e'', b'') = b''$, and so e'' is a left unit for B^{**} .

Conversely, let e'' be a left unit for B^{**} and suppose that $b \in B$. Then for every $b' \in B^*$, we have

$$\begin{aligned} \langle b', \pi(e_\alpha, b) \rangle &= \langle \pi^{***}(e_\alpha, b), b' \rangle = \langle e_\alpha, \pi^{**}(b, b') \rangle = \langle \pi^{**}(b, b'), e_\alpha \rangle \\ &= \langle e'', \pi^{**}(b, b') \rangle = \langle \pi^{***}(e'', b), b' \rangle = \langle b', b \rangle. \end{aligned}$$

Then we have $\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b'$ in B^* , and so by Cohen factorization theorem we are done. \square

Corollary 2-14. Let B be a Banach left A -module and A has a $LBAI$ $(e_\alpha)_\alpha \subseteq A$ such that $e_\alpha \xrightarrow{w^*} e''$ in A^{**} where e'' is a left unit for A^{**} . Suppose that $Z_{e''}^t(B^{**}) = B^{**}$. Then $\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b'$ in B^* if and only if e'' is a left unit for B^{**} .

For a Banach algebra A , we recall that a bounded linear operator $T : A \rightarrow A$ is said to be a left (resp. right) multiplier if, for all $a, b \in A$, $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$). We denote by $LM(A)$ (resp. $RM(A)$) the set of all left (resp. right) multipliers of A . The set $LM(A)$ (resp. $RM(A)$) is normed subalgebra of the algebra $L(A)$ of bounded linear operator on A .

Let B be a Banach left [resp. right] A -module and $T \in \mathbf{B}(A, B)$. Then T is called extended left [resp. right] multiplier if $T(a_1 a_2) = \pi_r(T(a_1), a_2)$ [resp. $T(a_1 a_2) = \pi_\ell(a_1, T(a_2))$] for all $a_1, a_2 \in A$.

We show by $LM(A, B)$ [resp. $RM(A, B)$] all of the Left [resp. right] multiplier extension from A into B .

Let $a' \in A^*$. Then the mapping $T_{a'} : a \rightarrow a'a$ [resp. $R_{a'} : a \rightarrow aa'$] from A into A^* is left [right] multiplier, that is, $T_{a'} \in LM(A, A^*)$ [$R_{a'} \in RM(A, A^*)$]. $T_{a'}$ is weakly compact if and only if $a' \in wap(A)$. So, we can write $wap(A)$ as a subspace of $LM(A, A^*)$.

Theorem 2-15. Let B be a Banach A -bimodule with a BAI $(e_\alpha)_\alpha \subseteq A$. Then

- (1) If $T \in LM(A, B)$, then $T(a) = \pi_r^{***}(b'', a)$ for some $b'' \in B^{**}$.
- (2) If $T \in RM(A, B)$, then $T(a) = \pi_\ell^{***}(a, b'')$ for some $b'' \in B^{**}$.

Proof. (1) Since $(T(e_\alpha))_\alpha \subseteq B$ is bounded, it has weakly limit point in B^{**} . Let $b'' \in B^{**}$ be a weakly limit point of $(T(e_\alpha))_\alpha$ and without loss generally, take $T(e_\alpha) \xrightarrow{w} b''$. Then for every $b' \in B^*$ and $a \in A$, we have

$$\langle \pi_r^{***}(b'', a), b' \rangle = \lim_{\alpha} \langle b', T(e_\alpha)a \rangle = \lim_{\alpha} \langle b', T(e_\alpha a) \rangle$$

$$= \lim_{\alpha} \langle T^*(b'), e_{\alpha} a \rangle = \langle T^*(b'), a \rangle = \langle b', T(a) \rangle.$$

It follows that $\pi_r^{***}(b'', a) = T(a)$.

(2) Proof is similar to (1).

□

In the proceeding theorem, if we take $B = A$, then we have the following statements

- (1) If $T \in LM(A)$, then $T(a) = a''a$ for some $a'' \in A^{**}$.
- (2) If $T \in RM(A)$, then $T(a) = aa''$ for some $a'' \in A^{**}$.

Let B be a Banach left [resp. right] A -module. Then for every $b \in B$, we define ℓ_b (resp. r_b) the linear mapping $a \rightarrow \pi_{\ell}(a, b)$ (resp. $a \rightarrow \pi_r(b, a)$).

In the proceeding theorem, for $T \in LM(A, B)$ [resp. $T \in RM(A, B)$] if B is weakly compact, then $T = r_b$ [resp. $T = \ell_b$], for some $b \in B$.

Let B be a Banach right A -module. Then, we define B^*B as a subspace of A^* including of all $\pi_r(b', b)$ for every $b' \in B^*$ and $b \in B$, that is, for every $a \in A$, we define

$$\langle \pi_r^*(b', b), a \rangle = \langle b', \pi_r(b, a) \rangle.$$

For Banach left A -module B , we also define $B^{**}B^*$ as a subset of A^* and for every $b'' \in B^{**}$, $b' \in B^*$ and $a \in A$, as follows

$$\langle \pi_{\ell}^{**}(b'', b'), a \rangle = \langle b'', \pi_{\ell}^*(b', a) \rangle.$$

A Banach space B is said to be weakly sequentially complete (WSC), if every weakly Cauchy sequence in B has a weak limit in B .

Definition 2-16. Let B be a Banach left A -module and $b'' \in B^{**}$. Suppose that $(b_{\alpha})_{\alpha} \subseteq B$ such that $b_{\alpha} \xrightarrow{w^*} b''$. We define the following set

$$\tilde{Z}_{b''}(A^{**}) = \{a'' \in A^{**} : \pi_{\ell}^{***}(a'', b_{\alpha}) \xrightarrow{w^*} \pi_{\ell}^{***}(a'', b'')\},$$

which is subspace of A^{**} . It is clear that $Z_{b''}(A^{**}) \subseteq \tilde{Z}_{b''}(A^{**})$, and so

$$Z_{B^{**}}(A^{**}) = \bigcap_{b'' \in B^{**}} Z_{b''}(A^{**}) \subseteq \bigcap_{b'' \in B^{**}} \tilde{Z}_{b''}(A^{**}).$$

For a Banach right A -module, the definition of $\tilde{Z}_{a''}^{t}(B^{**})$ is similar.

Let B be a Banach left A -module. Then we define $B^{**}B^*$ as follows.

$$B^{**}B^* = \{\pi_{\ell}^{**}(b'', b') : b'' \in B^{**} \text{ and } b' \in B^*\}.$$

It is clear that $B^{**}B^*$ is a subspace of A^* .

Let A be a Banach algebra. Then, A is said to be weakly sequentially complete ($= WSC$), if every weakly Cauchy sequence in A has a weak limit.

Theorem 2-17. Let B be a left Banach A -module and $T \in \mathbf{B}(A, B)$. Consider the following statements.

- (1) $T = \ell_b$, for some $b \in B$.
- (2) $T^{**}(a'') = \pi_\ell^{***}(a'', b'')$ for some $b'' \in B^{**}$ such that $\tilde{Z}_{b''}(A^{**}) = A^{**}$.
- (3) $T^*(B^*) \subseteq BB^*$.

Then (1) \Rightarrow (2) \Rightarrow (3).

Assume that B has WSC . If we take $T \in RM(A, B)$ and B has a sequential BAI , then (1), (2) and (3) are equivalent.

Proof. (1) \Rightarrow (2)

Let $T = \ell_b$, for some $b \in B$. Then $T^{**}(a'') = \ell_b^{**}(a'') = \pi_\ell^{***}(a'', b)$ for every $a'' \in A^{**}$, and so proof is hold.

(2) \Leftrightarrow (3)

Take $a'' \in (BB^*)^\perp$. Assume that $b'' \in B^{**}$ and $(b_\alpha)_\alpha \subseteq B$ such that $b_\alpha \xrightarrow{w^*} b''$. For every $b' \in B^*$, we have the following equality

$$\begin{aligned} \langle a'', T^*(b') \rangle &= \langle T^{**}(a''), b' \rangle = \langle \pi_\ell^{***}(a'', b''), b' \rangle = \lim_\alpha \langle \pi_\ell^{***}(a'', b_\alpha), b' \rangle \\ &= \lim_\alpha \langle a'', \pi_\ell^{**}(b_\alpha, b') \rangle = 0. \end{aligned}$$

It follows that $T^*(B^*) \subseteq BB^*$.

Take $T \in RM(A, B)$ and suppose that B is WSC with sequential BAI . It is enough, we show that (3) \Rightarrow (1). Assume that $(e_n)_n \subseteq A$ is a BAI for B . Then for every $b' \in B^*$, we have

$$\begin{aligned} |\langle b', T(e_n) \rangle - \langle b', T(e_m) \rangle| &= |\langle T^*(b'), e_n - e_m \rangle| = |\langle \pi_\ell^{**}(b, b'), e_n - e_m \rangle| \\ &= |\langle b, \pi_\ell^*(b', e_n - e_m) \rangle| = |\langle b', \pi_\ell(e_n - e_m, b) \rangle| \rightarrow 0. \end{aligned}$$

It follows that $(T(e_n))_n$ is weakly Cauchy sequence in B and since B is WSC , there is $b \in B$ such that $T(e_n) \xrightarrow{w} b$ in B . Let $a \in A$. Then for every $b' \in B^*$, we have

$$\begin{aligned} \langle b', \pi_\ell(a, b) \rangle &= \langle \pi_\ell^*(b', a), b \rangle = \lim_n \langle \pi_\ell^*(b', a), T(e_n) \rangle \\ &= \lim_n \langle b', \pi_\ell(a, T(e_n)) \rangle = \lim_n \langle b', T(ae_n) \rangle \\ &= \lim_n \langle T^*(b'), ae_n \rangle = \langle T^*(b'), a \rangle \\ &= \langle b', T(a) \rangle. \end{aligned}$$

Thus $\ell_b(a) = \pi_\ell(a, b) = T(a)$.

□

Example 2-18. Let G be a locally compact group. Then by convolution multiplication, $M(G)$ is a $L^1(G)$ -bimodule. Let $f \in L^1(G)$ and $T(\mu) = \mu * f$ for all $\mu \in M(G)$. Then $T^*(L^\infty(G)) \subseteq M(G)M(G)^*$. Also if we take $T(\mu) = f * \mu$ for all $\mu \in M(G)$, then we have $T^*(L^\infty(G)) \subseteq M(G)^*M(G)$.

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